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2000 J. Phys. A: Math. Gen. 33 3527

(http://iopscience.iop.org/0305-4470/33/17/4n2)

J. Phys. A: Math. Gen. **33** (2000) 3527–3529. Printed in the UK PII: S0305-4470(00)08987-3

COMMENT

Classical and quantum dynamics of a spin-$\frac{1}{2}$

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Received 26 November 1999, in final form 17 February 2000

Abstract. We reply to a comment on 'Semiclassical dynamics of a spin-$\frac{1}{2}$ in an arbitrary magnetic field'.

In a recent comment [1] Kochetov argues that our results [2] on the coherent state path integral for a spin-$\frac{1}{2}$ in an arbitrary magnetic field are based on a 'classical spin action inconsistent with the necessary boundary conditions'. Contrary from what is insinuated by the comment, our paper is not concerned with the *quantization* of a classical spin but solely with the *representation* of a *quantum* spin-$\frac{1}{2}$ in terms of a spin-coherent state path integral. Hence, the comment by Kochetov arguing primarily on a classical level is only vaguely relevant to our work and chiefly reconsiders the author's earlier work [3, 4] in the light of results in [2]. Indeed, the analysis in [2] allows for some conclusions of relevance to Kochetov's work as discussed below.

Before addressing the comment more specifically let us briefly reformulate the approach in [2]. We start with the two-dimensional Hilbert space, represented in the basis of spin-coherent states $|\Psi_g\rangle = \mathcal{D}^{1/2}(g)|\uparrow\rangle$, where $g \in SU(2)$ [5]. Since $|\Psi_h\rangle = \exp(\mathrm{i}\alpha)|\uparrow\rangle$ for an h in the maximal torus, $|\Psi_g\rangle$ and $|\Psi_{g'}\rangle$ describe the same physical state if there exists a $h \in U(1)$ such that $g' = gh$. Therefore, the group $SU(2)$ can be viewed as fibre bundle over the base manifold $SU(2)/U(1) \equiv S^2$ with fibre $U(1)$ [6] and the space of distinct spin-coherent states is canonically isomorphic to these left cosets. Parametrizing any $g \in SU(2)$ with Euler angles $(\vartheta, \varphi, \chi)$, we get

$$|\Omega\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{2}\chi}\mathrm{e}^{-\mathrm{i}\varphi S_z}\mathrm{e}^{-\mathrm{i}\vartheta S_y}|\uparrow\rangle. \qquad (1)$$

Here, the first factor on the right-hand side is just a phase factor and the rest determines the physical state. These states are not orthogonal but form an overcomplete basis in the Hilbert space. The overlap is readily evaluated and the identity may be represented as

$$I = \frac{1}{2\pi}\int \sin(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi|\Omega\rangle\langle\Omega|. \qquad (2)$$

Employing a Trotter decomposition, the propagator may be written as

$$\langle\Omega''|U(t)|\Omega'\rangle = \lim_{\epsilon\to 0} N\int\prod_{k=1}^{n}\sqrt{\det\omega_{ij}}\,\mathrm{d}\vartheta_k\,\mathrm{d}\varphi_k \times \exp\left\{\sum_{k=0}^{n}\left[\log\langle\Omega_{k+1}|\Omega_k\rangle + \frac{\mathrm{i}}{2}(\chi_{k+1}-\chi_k) - \mathrm{i}\epsilon\frac{\langle\Omega_{k+1}|H(k\epsilon)|\Omega_k\rangle}{\langle\Omega_{k+1}|\Omega_k\rangle}\right]\right\} \qquad (3)$$

where $\epsilon = t/n$, $(\Omega_0, \chi_0) = (\Omega', \chi')$, $(\Omega_{n+1}, \chi_{n+1}) = (\Omega'', \chi'')$. We are allowed to pass to the continuum limit if the paths stay continuous for $\epsilon \to 0$, which is not guaranteed if no Wiener measure occurs. Therefore, care must be taken in calculating the path integral [7–12]. To ensure an integration over continuous Brownian motion paths, we introduce a regularization by the spherical Wiener measure and are then allowed to write

$$\langle\Omega''|U(t)|\Omega'\rangle = \lim_{\nu\to\infty} N \int \prod_{s=0}^{t} \sqrt{\det \omega_{ij}}\, \mathrm{d}\vartheta(s)\, \mathrm{d}\varphi(s)$$
$$\times \exp\left\{ \mathrm{i} \int_0^t \mathrm{d}s \left[\frac{\mathrm{i}}{\nu}(g_{\vartheta\vartheta}\dot{\vartheta}^2 + g_{\varphi\varphi}\dot{\varphi}^2) + \theta_\vartheta \dot{\vartheta} + \theta_\varphi \dot{\varphi} - H(\vartheta, \varphi, s) \right] \right\}. \qquad (4)$$

Here, $N = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{1}{\pi}$ is a normalization factor, $g = \frac{1}{4}(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin(\vartheta)^2\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi)$ the metrical tensor, $\omega = \frac{1}{2}\sin(\vartheta)\, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$ the symplectic two-form of $SU(2)/U(1)$ [13] and $\theta = \frac{1}{2}(\cos(\vartheta)\, \mathrm{d}\varphi + \mathrm{d}\chi)$ its corresponding symplectic potential ($\omega = -\,\mathrm{d}\theta$).

Choosing in every left coset one special representant, i.e. fixing χ for every coherent state, one defines a section of the $SU(2)$ bundle. In particular, the choice $\chi = 0$ was adopted in [2]. It is important to note that *once χ has been fixed the symplectic potential is fixed as well* and manipulations of the form suggested by Kochetov [1] in equation (3) are no longer allowed. The very same reasoning applies in the parametrization used by Kochetov. Within the Gaussian decomposition [5] of the elements of $SU(2)$ by $g = z_- h z_+$ for $z_- \in Z_-$, $h \in U(1)$, $z_+ \in Z_+$ or equivalently $g = z_- b_+$ with $b_+ \in B_+$, we recognize that $\mathcal{D}^{1/2}(g)|\uparrow\rangle = \mathcal{D}^{1/2}(z_- h)|\uparrow\rangle$. Parametrizing z_- by the complex number ζ, the space of distinct spin coherent states is now isomorphic to elements in $SL(2,\mathbb{C})/B_+ \cong SU(2)/U(1)$. If the isomorphismus is defined explicitly by the spherical projection from the south pole of the sphere onto the complex plane, one has $\zeta = \tan(\frac{\vartheta}{2})\mathrm{e}^{\mathrm{i}\varphi}$, and makes use of a *different* section of the $SU(2)$ bundle by setting $\chi = -\varphi$. Therefore, a corresponding phase factor appears:

$$|\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}} \mathrm{e}^{\zeta S_-}|\uparrow\rangle = \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi}|\Omega\rangle. \qquad (5)$$

Again, with the choice $\chi = -\varphi$ there is no room for additional manipulations of the form (3) in [1]. While Kochetov's theory starts from a classical spin and employs geometric quantization to obtain a quantum propagator after an *ad hoc* modification of the symplectic potential, no such ambiguities arise if the representation of the quantum propagator in terms of a path integral is considered.

A main point in the critique by Kochetov [1] is the claim that the approach in [2] disagrees with boundary conditions in the classical limit. Since the physical states form a symplectic two-dimensional differential manifold with the closed two-form ω, the classical dynamics is determined by the Hamiltonian vector field $\omega(X_H, \cdot) = \mathrm{d}H$ which leads immediately to the classical equations of motion (23) in [2]. Note that in general there is *no classical path* connecting arbitrary but real boundary conditions $\bar{\Omega}(0) = \bar{\Omega}'$ and $\bar{\Omega}(t) = \bar{\Omega}''$. This is known as the 'overspecification problem'. Kochetov modifies the action to allow always for a 'classical' path, which is usually complex. If one applied the same rules to a simple harmonic oscillator there would be a 'classical' path connecting any initial phase space point (q', p') with any endpoint (q'', p''). This is clearly not what is usually meant by classical. Hence, the overspecification problem should *not* be removed in the classical limit. Yet, in the quantum problem, there is indeed a semiclassical path for any pair of real boundary conditions (see equations (23) *and* (24), (25) in [2]).

Finally, Kochetov believes that the exactness of the semiclassical propagator is 'obvious' and 'self-evident'. Replacing Kochetov's qualitative arguments by a more accurate treatment [14, 15], one finds the necessary condition $\theta(X_H) = H$ that θ is $SU(2)$ invariant and the stationary phase approximation becomes exact. For a Hamilton operator which is a linear combination of all three generators of the $SU(2)$ algebra we get three conditions which cannot be satisfied generally by fixing the phase χ appropriately. Therefore, there are no $SU(2)$-invariant potentials on S^2 and although $SU(2)$ is the group of isometric canonical transformations on the two-sphere [16], it does not preserve the sections. Hence, for magnetic fields of arbitrary time-dependence, the exactness of dominant stationary phase approximation (DSPA) is not 'self-evident', and prior to our work [2] it was rather expected that the DSPA would not provide a correct result [17].

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